Chapter 6, the last video. A summary of the Big Three:

Euclidean, Spherical, and Hyperbolic Geometries.

So, are they all axiomatic systems? YES! Do they have the same axioms? Nope.

Let's look at the parallel axiom as an example:

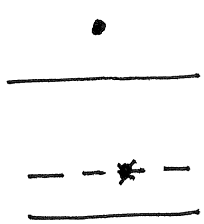
Euclidean:

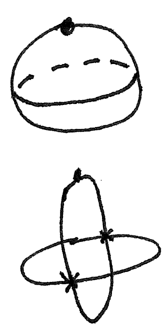
Exactly one line parallel through a given point external to the given line

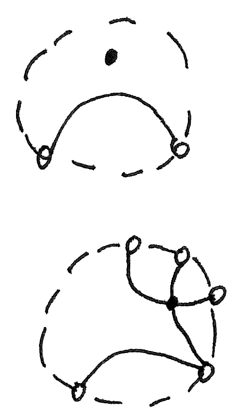
Spherical: No lines through the external point parallel to a given line

Hyperbolic: More than one line through an external point parallel to a given line plus two types of parallel lines: divergent and asymptotic.

Sketches:







Chapter 6 Popper Question1

All three geometries have undefined terms, axioms, definitions, and theorems.

A. True

B. False

Now let's talk undefined terms. The basic ones will be **point and line**. There are others of course.

Let's look at a summary for our models of those:

Euclidean: we've stuck basically to two dimensions, but of course 3D exists and is in the axioms. The Cartesian plane in 2D with y = mx+b style lines.

Spherical: We worked on the surface of the unit sphere with Euclidean 3D points (x, y, z). The formula is $x^2 + y^2 + z^2 = 1$. We reference the center of the sphere (0,0,0) often though it is not a point in our space. Note that ANY sphere could host a Spherical geometry on it's surface.

Hyperbolic: We looked at the Poincare Disc in the 2D plane. It is the unit disc, not including the unit circle. Formula for the points: $x^2 + y^2 < 1$. Be aware that there are other models, including 3D models.

So we have traditional Euclidean points for all 3. Just different sets and subsets of them.

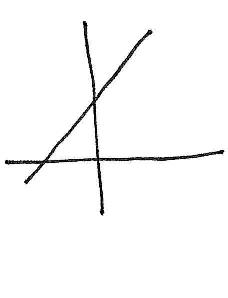
Now lines:

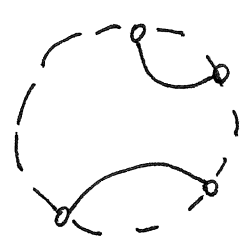
Euclidean, 2D…the good old y = mx+b

Spherical: Great Circles with the center in the plane of intersection

Hyperbolic: arcs of orthogonal circles

Sketches:

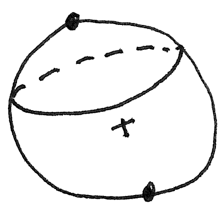




Now for **circles**: All three have them. All three use the SAME definition, too…but, of course there are real differences.

Euclidean, the usual.

Spherical, intersections of the surface and a plane that does not include the center of the sphere. The equator and great circles that include the north and south poles are LINES, not circles. Note that great care needs to be taken in identifying the center of the circle. There are two possible candidates. Chose the one with the shorter radius. Illustration:



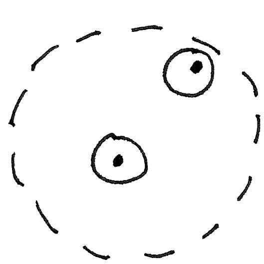
Chapter 6 Popper Question 2

The equator is the biggest circle in Spherical Geometry

A. True B. False

Hyperbolic, any Euclidean circle that is totally inside the interior of the unit disc.

Note that the center of the circle is more and more offset as you approach the boundary of the circle. This is a function of the distance formula (coming right up). Illustration:



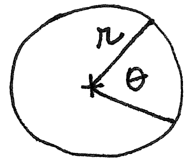
There is a **distance axiom** in each set of axioms. So you can measure distance in all three. Naturally, by now you expect this, the FORMULA is different in all three. Let's just look at those and inspect the differences.

Euclidean is the usual:

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

It can get infinitely large.

Spherical is D = r (theta) where theta is the central angle in radian measure. Let's look at that. Distance is bounded above and below. It cannot be larger than pi nor smaller than negative pi. This is seriously different from both Euclidean and Hyperbolic.



Hyperbolic, well let's just look at the formula. It's not for memorizing. Just note how different it is. It actually allows for infinite distances inside the unit circle!

$$\left| \ln \frac{(AM \cdot BN)}{(AN \cdot BM)} \right|$$

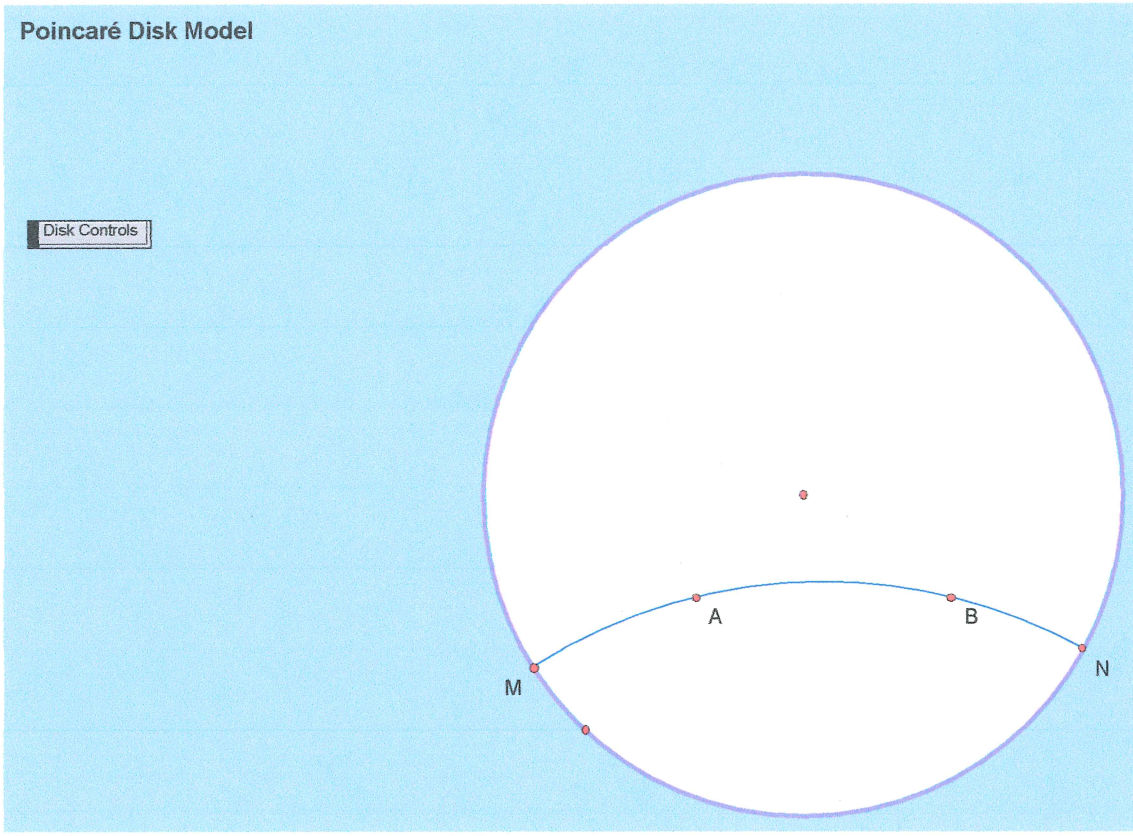
picture next page

AM NOT

BN not

in the space

in the space

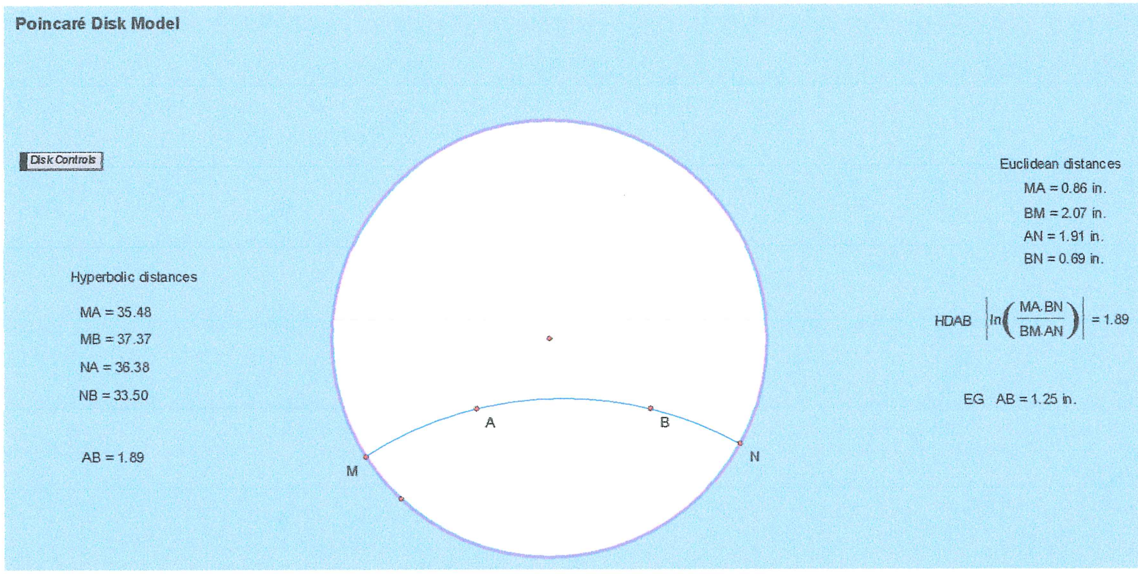


Euclidean measurements to calculate the Hyperbolic distance with a formula:

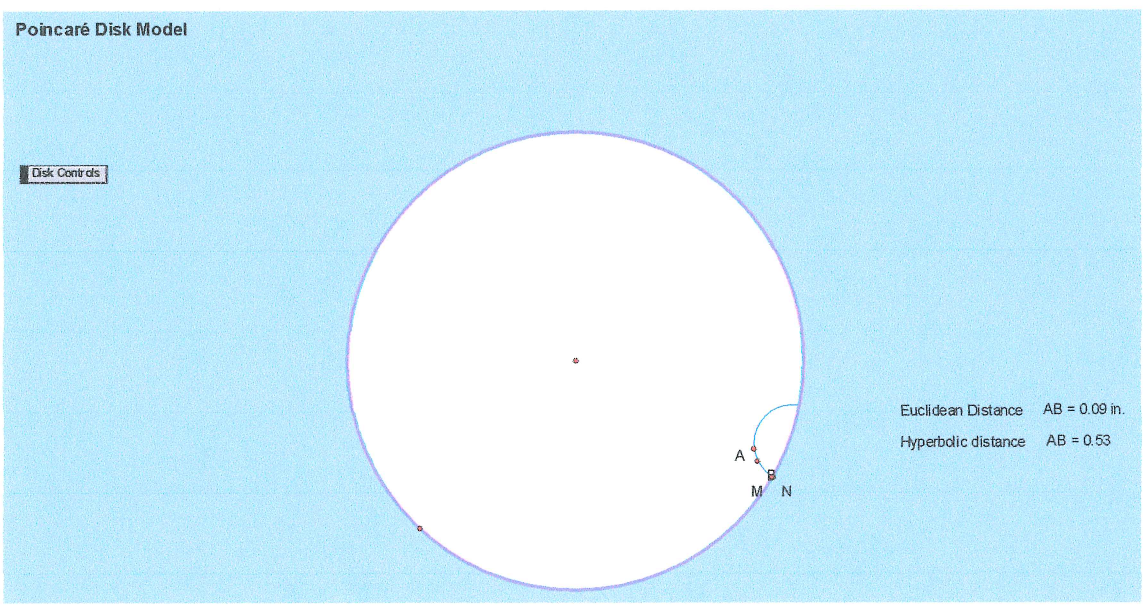
the Hdistance from A to B is $\left|\ln\left(\frac{AM \cdot BN}{AN \cdot BM}\right)\right|$ Note we are using points outside the space here, N and M. Just like in Spherical with the center of the circle.

Now this is a complicated formula but it works well. It can give infinitely large distances INSIDE the unit circle, in the disc. This results in some funny looking lengths along lines though. Let's look at a calculation and then at how distance appears in the center of the disc and over near the edge of the disc.

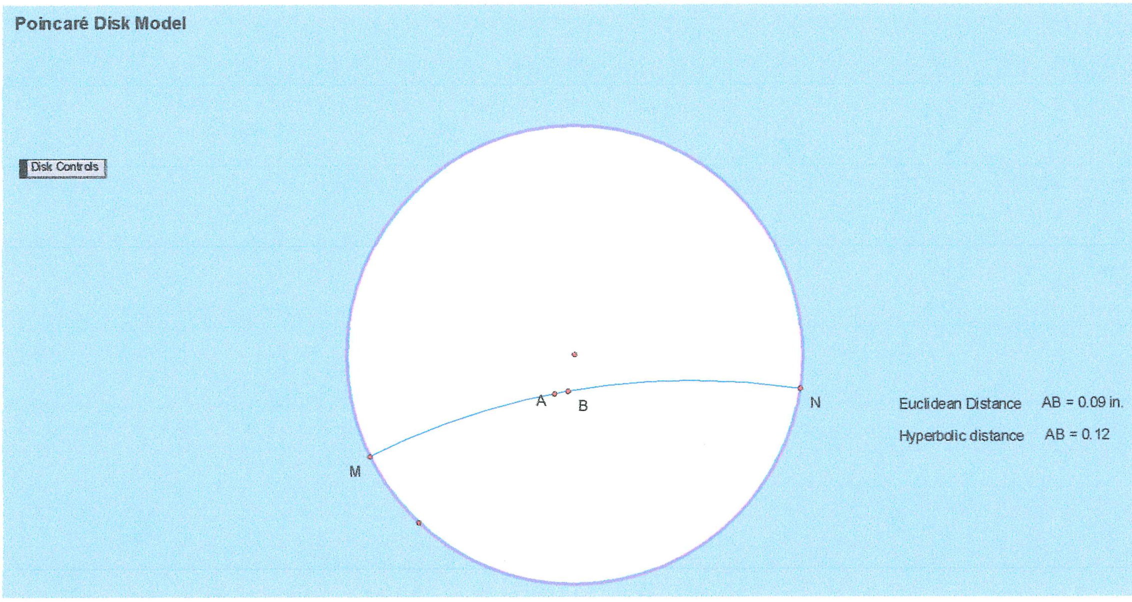
Just one calculation next page:



Distance in the center of the disc and at the edge:



Euclidean distance	.09
Hyperbolic distance	.52



Euclidean distance .09

Hyperbolic distance .12

Notice that the points in both pictures look the same distance apart to your Euclidean trained eyes, BUT once we run them through the Hyperbolic Distance Formula they are really different.

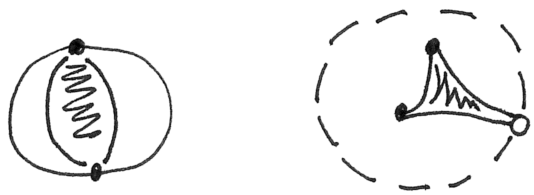
This is why circles have centers offset nearer the edge of the disc and more centered looking towards the center! This is straight from the formula. It actually can calculate HUGE distances inside the disc.

Chapter 6 Popper Question 3

EG and HG extend to infinity in their spaces. SG is bounded by pi and negative pi in distance measurements.

A. True B. False

Now let's look at **polygons**. Starting with Biangles. Both Spherical and Hyperbolic geometries have a polygon Euclidean doesn't. It's a shape with only two vertices. Let's look:



Now moving on to three sides: Triangles.

Euclidean triangles: the sum of the interior angles in 180 degrees all the time.

Spherical triangles: this sum varies and is always greater than 180 degrees.

Hyperbolic triangles: this sum varies and is always less than 180 degrees.

Note that there are equilateral triangles, isosceles triangles, and right triangles in all three geometries. In fact, in Spherical you can have a triangle with three right angles:

Chapter 6 Popper Question 4

There are no biangles in EG, but there are in HG and SG.

A. True

B. False

Now for one foray into quadrilaterals. All three geometries have them. Only Euclidean geometry has rectangles and squares though. Let's look at how that happens. The one type of quadrilateral that all three have is a Saccheri Quadrilateral. You take bases and erect perpendicular sides. So you have 3 sides. Then the difference show up right when you put on the summit, the top.

The summit angles are congruent in all 3, BUT

Euclidean: they are 90 degree, it's a rectangle

Spherical: they are obtuse, more than 90

Hyperbolic: they are acute, less than 90

And the summit lengths exhibit the same sort of differences:

Euclidean: the summit length equals the base length

Spherical: it is shorter than the base length

Hyperbolic: it is longer than the base length

Illustrations:

Now this type of quadrilateral (Saccheri or biperpendicular quadrilateral) is in all three. The OTHER types of quadrilaterals are there but each is unique to it's own space.

So here we are with three big spaces. We are locally, in Houston, Euclidean. From orbit, we can pop a Spherical geometry on the Earth. And once we are outside our solar system it is looking pretty Hyperbolic from the data getting sent to NASA.

There we go. The last popper and except for the final, we're done!